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An Alternation Theory for Copositive Approximation

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1. INTRODUCTION

In a recent series of papers, the problems of comonotone and copositive approximation have been investigated [6–8]. The primary focus of the above work has been to derive estimates for the error of the best constrained approximation analogous to the Jackson theorems of the standard theory. In one paper [7], however, it was noted that copositive approximation can be viewed as a special case of restricted range approximation [9], and from this existence and uniqueness of best copositive approximation was established. It remains to develop an alternation theory for this problem since no alternation theory was developed for the general case considered in [9].

2. BASIC DEFINITIONS AND NOTATION

Let M be an n-dimensional extended Chebyshev subspace of $C[a, b]$ of order 3 [3]. Let $f \in C[a, b]$ and define $K_f \subset M$ by

$$K_f = \{p \in M: p(x) f(x) \geqslant 0 \text{ for all } x \in [a, b]\}.$$

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Each function in K_f is said to be copositive with respect to f. If $p^* \in K_f$ has the property that

$$\|f - p^*\| = \inf_{p \in K_f} \|f - p\|,$$

where $\|h\| = \max\{|h(x)|: x \in [a, b]\}$, then we say that p^* is a best copositive approximation (from M) to f.

Let

$$L^0 = \{x \in [a, b]: f(x) < 0\}, \qquad L = \overline{L^0}$$

and

$$U^0 = \{x \in [a, b]: f(x) > 0\}, \qquad U = \overline{U^0},$$

where the bar denotes point set closure in the reals. Let $S = U \cap L$. If S contains more than n points, then K_f consists of just the zero function. We thus assume that S contains $m < n$ points. We will place one additional restriction on f that will guarantee that K_f consists of more than the zero function. First note that $t \in S$ if and only if $f(t) = 0$ and f changes sign at t. We shall say that f changes sign at $t \in (a, b)$ provided there exists $\eta > 0$ such that $(t, t + \eta] \cap U = \varnothing$ and $[t - \eta, t) \cap L = \varnothing$ (or $(t, t + \eta] \cap L = \varnothing$ and $[t - \eta, t) \cap U \varnothing$), and f does not vanish identically in any open interval containing t.

We shall say that f changes sign on the interval $[c, d]$, $a < c < d < b$, provided $t \in [c, d]$ implies $f(t) = 0$, and there exists an $\epsilon > 0$ such that for each $\eta > 0$, $\eta \leqslant \epsilon$, we have $\alpha \operatorname{sgn} f(x) \geqslant 0$, $x \in (c - \eta, c)$ and $\alpha \operatorname{sgn} f(x) \leqslant 0$, $x \in (d, d + \eta)$, with strict inequality holding for some $x' \in (c - \eta, c)$ and $x'' \in (d, d + \eta)$, where $\alpha = -1$ or $+1$. Thus, for example, f defined on $[-1, 1]$ by

$$\begin{aligned} f(x) &= -[(x + \tfrac{1}{2}) \sin(1/(x + \tfrac{1}{2})]^2, && -1 \leqslant x < -\tfrac{1}{2}, \\ &= 0, && -\tfrac{1}{2} \leqslant x \leqslant 0, \\ &= (x \sin(1/x))^2, && 0 < x \leqslant 1, \end{aligned}$$

changes sign on the interval $[-\frac{1}{2}, 0]$, whereas $\tilde{f}$ defined by $\tilde{f}(x) = f(x)$ for $x \in [-\frac{1}{2}, 1]$ and $\tilde{f}(x) = -f(x)$ for $x \in [-1, -\frac{1}{2})$ does not change sign on $[-\frac{1}{2}, 0]$. If f does not change sign on any interval and S contains $<n$ points, then we say that f is *admissible*. Thus, f defined above is not admissible, but $\tilde{f}$ is, since $S = \varnothing$.

In what follows we shall assume that f is admissible, which guarantees that K_f consists of nontrivial functions. This fact follows from the theory of extended Chebyshev systems, since given $m < n$ distinct points in (a, b), we can find an element $q \in M$ such that q has simple zeros at these points and no other zeros [3, p. 28]. It should be remarked that to just guarantee that

K_f contains a nontrivial function, it would suffice to assume that the number of elements in S plus the number of times f changes sign on intervals is strictly less than n. In this case, however, the alternation theory is much more difficult. Thus, we shall restrict our attention to the somewhat simpler case defined above.

Before describing our alternation theorems, it is worth remarking that these results differ from standard alternation theorems for constrained approximation (cf. [2, 5, 10], for example), in the sense that here they depend upon the best approximation. It is necessary, therefore, to spend some time introducing our results.

For $f \in C[a, b] \sim M$ and for fixed $p \in K_f$, $x \in [a, y]$ is said to be a *positive extreme point* for $f - p$ provided $f(x) - p(x) = \|f - p\|$, or $x \in U \sim S$ and $p(x) = 0$. Likewise, $x \in [a, b]$ is said to be a *negative extreme point* for $f - p$ provided $f(x) - p(x) = -\|f - p\|$, or $x \in L \sim S$ and $p(x) = 0$. Let X_p denote the set of all positive and negative extreme points for $f - p$. Note that X_p is a compact subset of $[a, b]$.

Now define σ on X_p by $\sigma(x) = +1$ if x is a positive extreme point and $\sigma(x) = -1$ if x is a negative extreme point. σ is well defined, since the set of positive extremals and the set of negative extremals are disjoint.

Next, we define a new type of signlike function, $\mathrm{sg}(f(x))$, for f at each $x \in [a, b]$ as follows. Set $\mathrm{sg}(f(x)) = 0$ if $x \in S$ and $\mathrm{sg}(f(x)) = \mathrm{sgn}(f(x))$ if $f(x) \neq 0$. If $f(x) = 0$ and $x \notin S$, then, since we are assuming that f is admissible and $f \not\equiv 0$, there exists $\rho > 0$ such that either $(x - \rho, x + \rho) \cap L = \varnothing$ and $(x - \rho, x + \rho) \cap U \neq \varnothing$, or $(x - \rho, x + \rho) \cap L \neq \varnothing$ and $(x - \rho, x + \rho) \cap U = \varnothing$. Since these two possibilities are mutually exclusive, we may define $\mathrm{sg}(f(x)) = 1$ if the first holds and $\mathrm{sg}(f(x)) = -1$ if the second holds. Observe that $\mathrm{sg}(f(x)) \neq 0$ for $x \notin S$, $y \in L \sim S$ implies $\mathrm{sg}(f(x)) = -1$, and $x \in U \sim S$ implies $\mathrm{sg}(f(x)) = 1$. Also, if $x \in X_p$ and $\mathrm{sg}(f(x))\, \sigma(x) = -1$, then $|f(x)| < |p(x)|$ must hold. Indeed, consider the case $\mathrm{sg}(f(x)) = 1$, $\sigma(x) = -1$. Here we must have either $f(x) - p(x) = -\|f - p\|$ or $x \in L \sim S$ and $p(x) = 0$. In the first case $p(x) = f(x) + \|f - p\|$, so that $p(x) > f(x) \geqslant 0$, while in the second case we must also have $\mathrm{sg}(f(x)) = -1$, violating our assumption. Thus, the second possibility cannot occur. Likewise, one can establish that $0 \leqslant f(x) < p(x)$ when $\mathrm{sg}(f(x)) = -1$, $\sigma(x) = 1$.

Let $x, y \in X_p$, $x < y$, $(x, y) \cap X_p = \varnothing$, and $(x, y) \cap S = \{z_{i+1}, z_{i+2}, \ldots, z_{i+\nu}\}$. $\nu \geqslant 0$, where $\nu = 0$ implies that $(x, y) \cap S = \varnothing$. (Remark: (x, y) denotes the open interval with endpoints x and y.) Under our assumption that f is admissible, we shall say that $f - p$ alternates once between x and y (or in (x, y)) if $\sigma(x) = (-1)^{\nu+1} \sigma(y)$. $f - p$ is said to alternate twice between x and y if $\mathrm{sg}(f(x))\, \sigma(x) = -1$, $\sigma(x) = (-1)^{\nu} \sigma(y)$, and there exists at least one $z_j \in (x, y) \cap S$ with $p'(z_j) = 0$. Note that in this case we must also have $\mathrm{sg}(f(y))\,\sigma(y) = -1$, $|f(y)| < |p(y)|$. In addition, $f - p$ is said to alternate once in each of the following two cases:

(i) On (a, y) if $y \in X_p$, $[a, y) \cap X_p = \varnothing$, $\mathrm{sg}(f(y))\,\sigma(y) = -1$, and p has at least $\nu + 1$ zeros in $[a, y]$ counting multiplicities up to order 2, where $[a, y] \cap S = \{z_1, \ldots, z_\nu\}$.

(ii) On (x, b) if $x \in X_p$, $(x, b] \cap X_p = \varnothing$, $\mathrm{sg}(f(x))\,\sigma(x) = -1$, and p has at least $\nu + 1$ zeros in $[x, b]$ counting multiplicites up to order 2, where $[x, b] \cap S = \{z_{m-\nu+1}, \ldots, z_m\}$.

Note that if $z \in S$ and $p'(z) = 0$, then we must also have that $p''(z) = 0$, since $M \subset C^2[a, b]$ and p changes sign at z.

We say that the set of open intervals $\{(x_i, y_i)\}_{i=1}^{\mu}$ is an *alternant of length r* for $f - p$ provided $y_i \leqslant x_{i+1}$ for $i = 1, 2, \ldots, \mu - 1$, $f - p$ alternates w_i times on (x_i, y_i), where $w_i = 1$ or 2 as defined above, and $\sum_{i=1}^{\mu} w_i = r$.

Finally, we should make a couple of remarks about some of the properties of the extended Chebyshev subspace, M, of $C[a, b]$, of order 3. First of all, $M \subset C^2[a, b]$, and $p \in M$ is said to have $x \in [a, b]$ as a zero of order ν, $\nu = 1, 2$, if $p^{(j)}(x) = 0$, $0 \leqslant j \leqslant \nu - 1$, and $p^{\nu}(x) \neq 0$. If $x \in [a, b]$ is such that $p^{(j)}(x) = 0$ for $j = 0, 1, 2$, then we say that x is a zero of order (at least) 3. Counting zeros of order ν, $\nu = 1, 2, 3$, as ν zeros, we have that each nonzero function $p \in M$ can have at most $n - 1$ zeros.

3. Main Results

In this section we wish to develop an alternation theory for the copositive approximation of admissible functions. In what follows, let $S = \{z_1, z_2, \ldots, z_m\}$, where $m < n$.

THEOREM 1. *Let $f \in C[a, b] \sim M$ be an admissible function. Then $p \in K_f$ is a best copositive approximation to f if and only if there exists a set of open intervals $\{(x_i, y_i)\}_{i=1}^{\mu}$ which is an alternant of length $n - m$ for $f - p$.*

Proof($\Leftarrow$). Suppose there exists $q \in K_f$ for which $\|f - q\| < \|f - p\|$. Under this assumption we shall prove that $p - q$ has at least n zeros, counting multiplicities up to order 3.

Fix i, $i = 1, 2, \ldots, \mu$, and consider the open interval (x_i, y_i), where $(x_i, y_i) \cap S = \{z_{i+1}, \ldots, z_{i+\nu}\}$, $\nu \geqslant 0$.

LEMMA 1. (a) *If $p(x_i) = q(x_i) = 0$ (or $p(y_i) = q(y_i) = 0$), then $p - q$ has at least $\nu + w_i + 1$ zeros in $[x_i, y_i]$.*

(b) *If* $p(x_i) \neq q(x_i), p(y_i) \neq q(y_i)$ *and* $w_i = 1$, *then* $p - q$ *has at least* $\nu + w_i$ *zeros in* (x_i, y_i).

(c) *If* $w_i = 2$, *then* $p - q$ *has at least* $\nu + w_i$ *zeros in* (x_i, y_i).

Proof. (a) In this case w_i must equal 1 and neither p nor q can change sign at x_i. Therefore, $p'(x_i) = q'(x_i) = 0$, so that $p - q$ has at least $\nu + 2 = \nu + w_i + 1$ zeros in $[x_i, y_i]$.

(b) Let us consider the case where $\sigma(x_i) = -1$ and ν is odd. In this case we must have $\sigma(y_i) = -1$, $p(x_i) > q(x_i)$, and $p(y_i) > q(y_i)$. Now, if $p - q$ has only $z_{i+1}, \ldots, z_{i+\nu}$ as simple zeros, then $(p - q)(x_i) > 0$ and ν odd imply that $(p - q)(y_i) < 0$. Thus, $p - q$ must have at least one of $z_{i+1}, \ldots, z_{i+\nu}$ as a zero of order at least two, or another zero in (x_i, y_i) different from $z_{i+1}, \ldots, z_{i+\nu}$. The other cases follow by similar arguments. For the two special cases where $w_i = 1$ on $[a, y_1]$ with $[a, y_1] \cap X_p = \varnothing$ or $w_i = 1$ on $[x_\mu, b]$ with $(x_\mu, b] \cap X_p = \varnothing$, the desired result follows from essentially the same argument as that given in part (c) to follow.

(c) Since we must have $\operatorname{sg}(f(x_i))\,\sigma(x_i) = \operatorname{sg}(f(y_i))\,\sigma(y_i) = -1$ in this case, we have that $|f(x_i) - p(x_i)| = |f(y_i) - p(y_i)| = \|f - p\|$, so that $|p(x_i)| > |q(x_i)|$ and $|p(y_i)| > |q(y_i)|$. Let z_j be the first element of $(x_i, y_i) \cap S$ for which $p'(z_j) = 0$. Now suppose that $p - q$ vanishes in (x_i, z_j) only at $z_{i+1}, \ldots, z_{j-1}$, and that each of these is a simple zero (set is empty if $j = i + 1$). Then, by considering two cases $(0 \geqslant f(x) > p(x)$ and $q(x) > p(x)$, or $0 \leqslant f(x) < p(x)$ and $q(x) < p(x))$, we have that $|p(t)| > |q(t)|$ must hold in some interval of the form $(z_j - \rho, z_j)$, $\rho > 0$. Since $p, q \in C^2[a, b]$, this implies that $q'(z_j) = 0$. However, since both p and q change sign at z_j, we must also have that $p''(z_j) = q''(z_j) = 0$. Thus, $p - q$ has z_j as a zero of order at least 3 and our desired result follows.

Thus, assume that $p - q$ has precisely one additional zero in $[x_i, z_j)$. That is, $p - q$ either has one simple zero in $[x_i, z_j) \sim \{z_{i+1}, \ldots, z_{j-1}\}$ or has one of $z_{i+1}, \ldots, z_{j-1}$ as a double zero. This implies that the inequality $|q(x)| > |p(x)|$ holds for all $x \in (z_j - \epsilon, z_j)$, for some $\epsilon > 0$.

Now, if $p - q$ has only $z_{j+1}, \ldots, z_{i+\nu}$ as simple zeros in $(z_j, y_i]$ and has no other zeros, then we must have that $|q(y_i)| > |p(y_i)|$. But this would be a contradiction. Hence, $p - q$ must have an additional zero in $(z_j, y_i]$, proving that $p - q$ has at least $\nu + w_i$ zeros in (x_i, y_i). The proof of Lemma 1 is complete.

We now count the total number of zeros of $p - q$ on $[a, b]$. Assume that there are η elements of S in the interior of the intervals of type (a) in Lemma 1, δ elements of S in the interior of the intervals of types (b) and (c), and $m - \delta - \eta$ elements of S in the rest of $[a, b]$. It is now easy to observe that $p - q$ has at least

(i) $\eta + \sum_{(a)} w_i$ zeros in the union of intervals of type (a);

(ii) $\delta + \sum_{(b),(c)} w_i$ zeros in the union of intervals of types (b) and (c);

(iii) $m - \delta - \eta$ zeros on the rest of $[a, b]$.

Thus, $p - q$ has at least $\eta + \delta + m - \delta - \eta + \sum_{i=1}^{\mu} w_i = n$ zeros on $[a, b]$. Hence $p = q$, and the proof of this part of the theorem is complete.

($\Rightarrow$) Suppose $f \notin M$ and $p \in K_f$ is a best copositive approximation to f. Assume that $\{(x_i, y_i)\}_{i=1}^{\mu}$ is an alternant for $f - p$ with $\sum_{i=1}^{\mu} w_i = k < n - m$, where k is maximal. We will construct a new function $r \in K_f$, for which $\|f - r\| < \|f - p\|$, thus contradicting our assumption that $k < n - m$. In what follows we shall assume that $S \neq \varnothing$, since for $S = \varnothing$, this problem reduces to a special case of restricted range approximation, [10], for which the desired alternation is known to hold.

The assumption that k is maximal is easily seen to require that for each $i = 1, 2,..., \mu - 1$, there are no alternations in $[y_i, x_{i+1}]$. Specifically, for each $x \in (y_i, x_{i+1}] \cap X_p$ with $[y_i, x] \cap S = \{z_{i+1},..., z_{i+\nu}\}$, $\nu \geqslant 0$, we must have that $\sigma(y_i) = (-1)^{\nu} \sigma(x)$. Furthermore, if there exists $z_j \in [y_i, x] \cap S$ with $p'(z_j) = 0$, then we must also have that $|p(y_i)| \leqslant |f(y_i)|$ and $|p(x)| \leqslant |f(x)|$. Also, if $x_1 > a$, then for each $x \in [a, x_1) \cap X_p$ with $[x, x_1] \cap S = \{z_{i+1},..., z_{i+\nu}\}$, we must have $\sigma(x_1) = (-1)^{\nu} \sigma(x)$. Also, if there exists $z_j \in [x, x_1] \cap S$ with $p'(z_j) = 0$, then we must have $|p(x_1)| \leqslant |f(x_1)|$ and $|p(x)| \leqslant |f(x)|$. Finally, for this case we must also have that, if there exists $z_j \in [a, x_1] \cap S$ with $p'(z_j) = 0$, then $|p(x_1)| \leqslant |f(x_1)|$. Likewise, if $y_\mu < b$, then similar statements are true for the interval $[y_\mu, b]$.

We begin the proof with the construction of a set of $k + m$ distinct points in (a, b) and a function $q \in M$. Consider the interval (x_i, y_i), for $i = 1,..., \mu$. If $w_i = 1$, define a point $s_i \in (x_i, y_i)$ as follows. First consider the case where $(x_i, y_i) \cap S = \varnothing$. If $p(t) \neq 0$ for all $t \in (x_i, y_i)$, set $s_i = (x_i + y_i)/2$. If $p(t) = 0$ for some $t \in (x_i, y_i)$, then set $t_i' = \min\{t \in (x_i, y_i) : p(t) = 0\}$ and $t_i'' = \max\{t \in (x_i, y_i) : p(t) = 0\}$. Now, if $x_i \in X_p$ and $\operatorname{sg}(f(x_i))\, \sigma(x_i) = 1$, set $s_i = (t_i'' + x_i)/2$, and if $x_i \in X_p$ and $\operatorname{sg}(f(x_i))\, \sigma(x_i) = -1$, set $s_i = (t_i' + y_i)/2$. If $x_i \notin X_p$ (i.e., $i = 1$, $w_1 = 1$, $x_1 = a$ with $[a, y_1) \cap X_p = \varnothing$, $\operatorname{sg}(f(y_1))\, \sigma(y_1) = -1$, and p has at least $\nu + 1$ zeros in $[a, y_1]$ counting multiplicities up to order 2, where $[a, y_1] \cap S = \{z_1,..., z_\nu\}$), define t_i'' as before and set $s_1 = (t_1'' + y_1)/2$. Next, consider the case that $(x_i, y_i) \cap S = \{z_{i+1},..., z_{i+\nu}\}$. Define t_i' and t_i'' as before, and note that $t_i' \leqslant z_{i+1}$ and $t_i'' \geqslant z_{i+\nu}$. Now, if $\operatorname{sg}(f(x_i))\, \sigma(x_i) = 1$, set $s_i = (t_i'' + y_i)/2$, whereas, if $\operatorname{sg}(f(x_i))\, \sigma(x_i) = -1$, set $s_i = (t_i' + x_i)/2$. Observe that in the case where $|f(x_i)| < |p(x_i)|$ and there exists $z_j \in (x_i, y_i) \cap S$ with $p'(z_j) = 0$, we must have that $|f(y_i)| > |p(y_i)|$, since $w_i = 1$. Finally, consider the case where $w_i = 2$. In this case we must have that $(x_i, y_i) \cap S = \{z_{i+1},..., z_{i+\nu}\}$, $\nu \geqslant 1$, with $p'(z_j) = 0$ for at least one z_j, $i + 1 \leqslant j \leqslant i + \nu$, $|f(x_i)| <$

$|p(x_i)|$, and $|f(y_i)| < |p(y_i)|$. Define t_i', t_i'' as before and set $s_i' = (t_i' + x_i)/2$ and $s_i = (t_i'' + y_i)/2$.

Let T denote the set of all the points constructed above, and set $Z = T \cup S$. Note that Z consists of precisely $k + m < n$ distinct points. Since M is an extended Chebyshev system of order 3, it follows that there exists $q \in M$, such that q has each point of Z as a simple zero and q vanishes only at these points [3]. We shall show that there exists $\epsilon > 0$, such that $r_\epsilon = p + \epsilon q$ is copositive with f and $\|f - r_\epsilon\| < \|f - p\|$, where q satisfies the requirement that sgn $q(y_1) = \sigma(y_1)$.

Let us first show that sgn $q(y_i) = \sigma(y_i)$ and sgn $q(x_i) = \sigma(x_i)$ for $i = 1,..., \mu$, provided $x_1 \in X_p$ and $y_\mu \in X_p$. (That is, for the special case when $x_1 = a$, $[a, y_1) \cap X_p = \varnothing$, with $w_1 = 1$, we do not necessarily have that $\sigma(a) =$ sgn $q(a)$. Likewise, for the special case when $y_\mu = b$ and $(x_\mu, b] \cap X_p = \varnothing$). Indeed, suppose $x_1 \in X_p$ and $(x_1, y_1) \cap S = \{z_{i+1},..., z_{i+\nu}\}$, $\nu \geqslant 0$. Now, if $w_i = 1$, then $\sigma(x_1) = (-1)^{\nu+1} \sigma(y_1)$ and, in this case, q has simple zeros at s_1, $z_{i+1},..., z_{i+\nu}$ and only at these points in $[x_1, y_1]$. Thus, sgn $q(x_1) = (-1)^{\nu+1}$ sgn $q(y_1)$, so that sgn $q(x_1) = \sigma(x_1)$, as desired. If $w_1 = 2$, then $\sigma(x_1) = (-1)^\nu \sigma(y_1)$ and sgn $q(x_1) = (-1)^{\nu+2}$ sgn $q(y_1)$, since q has simple zeros at s_1', $z_{i+1},..., z_{i+\nu}$, s_1 in $[x_1, y_1]$, once again establishing the desired result. Now consider $q(x_2)$. Let $(y_1, x_2) \cap S = \{z_{i+1},..., z_{i+\nu}\}$, $\nu \geqslant 0$. Then, since k is maximal, we must have that $\sigma(x_2) = (-1)^\nu \sigma(y_1)$. From this the desired result immediately follows, since q has only $z_{i+1},..., z_{i+\nu}$ as simple zeros in $[y_1, x_2]$. Now one can show that $\sigma(x_2) =$ sgn $q(x_2)$ implies $\sigma(y_2) =$ sgn $q(y_2)$ with the same argument used in the (x_1, y_1) case, and the remaining cases follow immediately.

Now let us consider the interval $[y_i, x_{i+1}]$ for fixed i, $i = 1,..., \mu - 1$. We shall first show that there exists $\epsilon_0 > 0$ such that for each ϵ, $0 < \epsilon \leqslant \epsilon_0$, $\max_{x\in[y_i, x_{i+1}]} |f(x) - r_\epsilon(x)| < \|f - p\|$. Indeed, since sgn $q(y_i) = \sigma(y_i)$ and $f - p$ does not alternate on $[y_i, x_{i+1}]$, we must have that for each $x \in [y_i, x_{i+1}] \cap X_p$, $\sigma(x) =$ sgn $q(x)$. Thus, if $[y_i, x_{i+1}] \cap S = \{z_{i+1},..., z_{i+\nu}\}$ and we set $t_0 = y_i$, $t_j = z_{i+j}$, $j = 1,..., \nu$, $t_{\nu+1} = x_{i+1}$, then for any $t \in [t_j, t_{j+1}]$ we have that $\sigma(y_i)(-1)^j q(t) \geqslant 0$, and, for any $x \in [t_j, t_{j+1}] \cap X_p$, $\sigma(x) = (-1)^j \sigma(y_i)$. Without loss of generality, assume $\sigma(x) = -1$. Then, for all $t \in [t_j, t_{j+1}]$, we must have that $f(t) - p(t) < \|f - p\|$, since k is maximal. Also, for $\epsilon > 0$, we have for $t \in [t_j, t_{j+1}]$ that $f(t) - r_\epsilon(t) = f(t) - p(t) - \epsilon q(t) > f(t) - p(t) \geqslant -\|f - p\|$. Thus, from continuity and compactness considerations, there exists an $\epsilon_j > 0$ such that for each ϵ, $0 < \epsilon \leqslant \epsilon_j$, we have $\max_{t\in[t_j, t_{j+1}]} |f(t) - r_\epsilon(t)| < \|f - p\|$. Repeating this argument for each of these subintervals and letting $\epsilon_0 = \min \epsilon_j$, we have our desired result on $[y_i, x_{i+1}]$.

Next, we must show that there exists $\epsilon_1 > 0$ such that for each ϵ, $0 < \epsilon \leqslant \epsilon_1$, r_ϵ is copositive with f on $[y_i, x_{i+1}]$. Note that both f and q

change sign in $[y_i, x_{i+1}]$ at the points of $[y_i, x_{i+1}] \cap S$. Thus, either q or $-q$ is copositive with f on $[y_i, x_{i+1}]$. (Actually, if $f \equiv 0$ on $[y_i, x_{i+1}]$, then both are copositive with f on $[y_i, x_{i+1}]$.) Let us first consider the case where there exists $z_j \in [y_i, x_{i+1}] \cap S$ with $p'(z_j) = 0$. Since k is maximal, we have that $|p(y_i)| \leqslant |f(y_i)|$. We claim that in this case f is copositive with q. Indeed, suppose $\sigma(y_i) = -1$; then either $f(y_i) - p(y_i) = -\|f - p\|$ or $p(y_i) = 0$ and $y_i \in L$. If the latter case occurs, then clearly f and q are copositive as $\operatorname{sgn} q(y_i) = -1$. Thus, suppose $f(y_i) - p(y_i) = -\|f - p\|$. Then $p(y_i) = \|f - p\| + f(y_i) > f(y_i)$. Since we must also have $|p(y_i)| \leqslant |f(y_i)|$, it follows that $f(y_i) < 0$, and our desired result follows. The case where $\sigma(y_i) = +1$ is proved in the same manner. Thus, in this case r_ϵ and f are copositive on $[y_i, x_{i+1}]$ for any $\epsilon > 0$. Finally, let us consider the case where $[y_i, x_{i+1}[\cap S = \{z_{i+1}, ..., z_{i+\nu}\}$, $\nu \geqslant 0$, and for which $p'(z_j) \neq 0$ for each z_j in the above intersection. Furthermore, let us assume that f and q are *not* copositive on $[y_i, x_{i+1}]$, so that f and $-q$ *are* copositive. Now suppose that $x \in (L \cup U) \cap ([y_i, x_{i+1}] \sim S)$. Then we claim that $p(x) \neq 0$. Indeed, suppose $x \in L \cap ([y_i, x_{i+1}] \sim S)$ and $p(x) = 0$. Then $x \in X_p$ and $\sigma(x) = -1$. Also, $x \in L \sim S$ implies that $q(x) > 0$, since $-q$ is copositive with f. But this contradicts the fact that $\sigma(x) = \operatorname{sgn} q(x)$ for all $x \in [y_i, x_{i+1}] \cap X_p$. Thus, we have that p and q both vanish at only the points of S in $\Gamma = [y_i, x_{i+1}] \cap (L \cup U)$ and that they both change sign at these points. Also, $\operatorname{sgn} p(x) = -\operatorname{sgn} q(x)$ for each $x \in \Gamma$. Now, at each $z_j \in \Gamma \cap S$ we have that $p'(z_j) \neq 0$. Thus, there exists $\theta_j \leqslant \frac{1}{2} \min\{z_{i+1} - z_i : i = 1, ..., m\}$, $\theta_j > 0$, such that $p'(x) \neq 0$ in $I_j = [z_j - \theta_j, z_j + \theta_j]$. Since $p(x) = p'(\xi_x)(x - z_j)$ and $q(x) = q'(\delta_x)(x - z_j)$ for each $x \in I_j$, where ξ_x and δ_x are between x and z_j, we can select $\epsilon_j > 0$ such that $0 < \epsilon \leqslant \epsilon_j$ implies that $|p(x)| \geqslant \epsilon |q(x)|$ for all $x \in I_j$. Repeat this argument for each $z_j \in \Gamma \cap S$ and $\epsilon' = \min \epsilon_j$. Then $\Gamma \sim (\bigcup_j (z_j - \theta_j, z_j + \theta_j))$ is a compact (possibly empty) subset of $[a, b]$ and $p(x) \neq 0$ for each x in this subset. Thus, we can find $\epsilon_1 > 0$, $\epsilon_1 \leqslant \epsilon_0$, such that $0 < \epsilon \leqslant \epsilon_1$ implies that $|p(x)| \geqslant \epsilon |q(x)|$ on this set. Thus, we have for $0 < \epsilon \leqslant \epsilon_1$ that r_ϵ is copositive with p on $[y_i, x_{i+1}] \cap (L \cup U)$, and hence copositive with f on $[y_i, x_{i+1}]$.

Next, we wish to consider an interval of the form $[x_i, y_i]$, i fixed, $i = 1, ..., \mu$. First of all, select $\delta > 0$ such that q does not vanish on $\Gamma_1 = [x_i, x_i + \delta] \cup [y_i - \delta, y_i]$. Then, since $f - p$ does not alternate on either $[x_i, x_i + \delta]$ or $[y_i - \delta, y_i]$, $\sigma(y_i) = \operatorname{sgn} q(y_i)$ and $\sigma(x_i) = \operatorname{sgn} q(x_i)$, we can find $\epsilon_2 > 0$ such that $0 < \epsilon \leqslant \epsilon_2$ implies $\max_{x \in \Gamma_1} |f(x) - r_\epsilon(x)| < \|f - p\|$, by precisely the same continuity and compactness arguments given earlier. Also, since $(x_i, y_i) \cap X_p = \varnothing$, we have that $|f(x) - p(x)| < \|f - p\|$ for all $x \in [x_i + \delta, y_i - \delta]$. Thus, there exist $\epsilon_i > 0$, $\epsilon_i < \epsilon_2$, such that $0 < \epsilon \leqslant \epsilon_i$ implies $\max_{x \in [x_i + \delta, y_i - \delta]} |f(x) - r_\epsilon(x)| < \|f - p\|$. Note that this is also true if $x_1 = a$ and $[a, y_1) \cap X_p = \varnothing$, or $y_\mu = b$ and $(x_\mu, b] \cap X_p = \varnothing$.

Thus, all that remains is to prove that for $\epsilon > 0$ sufficiently small, r_ϵ is copositive with f on $[x_i, y_i]$. Let us first note that if $w_i = 1$ and $\text{sg}(f(x_i))\,\sigma(x_i) = 1$, then $\text{sg}(f(x_i))\,\text{sgn}\,q(x_i) = 1$, so that f and q are copositive in $[x_i, s_i]$ (where $s_i > z_j$ for each $z_j \in [x_i, y_i] \cap S$), implying that r_ϵ and f are copositive in $[x_i, s_i]$ for any $\epsilon > 0$. Now consider the interval $(s_i, y_i]$. By our construction, p does not vanish in $[s_i, y_i)$. Also, from $\sigma(x_i) = (-1)^{\nu+1}\sigma(y_i)$ ($\nu = 0$ if $[x_i, y_i] \cap S = \varnothing$) and $\text{sg}(f(x_i)) = (-1)^\nu\,\text{sg}(f(y_i))$, we have that $\text{sg}(f(y_i))\,\sigma(y_i) = -1$. Hence $|p(y_i)| > |f(y_i)|$, implying $p(y_i) \neq 0$. Now either f vanishes identically in $[s_i, y_i]$, so that r_ϵ and f are copositive in $[s_i, y_i]$ for any choice of $\epsilon > 0$, or there exists $t \in (s_i, y_i)$ such that $f(t) \neq 0$. Since p has no zeros in $[s_i, y_i]$ by construction, we must have that $p(t) f(t) > 0$ holds, and hence we can choose $\alpha_0 > 0$ such that for any ϵ, $0 < \epsilon < \alpha_0$, r_ϵ and f will be copositive in $[s_i, y_i]$ (e.g., let $\alpha_0 = \frac{1}{2}(\min_{t\in[s_i,y_i]} |p(t)|/\max_{t\in[s_i,y_i]} |q(t)|)$, so that $\text{sgn}\,r_\epsilon(x) = \text{sgn}\,p(x)$ in $[s_i, y_i]$).

Similarly, if $w_i = 1$ and $\text{sg}(f(x_i))\,\sigma(x_i) = -1$, we have that $s_i < z_j$ for all $z_j \in [x_i, y_i] \cap S$, and f and q are copositive in $[s_i, y_i]$. Since $|f(x_i)| < |p(x_i)|$ must hold, we can show as above that f and r_ϵ are copositive on $[x_i, s_i]$ for sufficiently small positive ϵ.

Finally, the same argument can be given for the case that $x_1 = a$, $[a, y_1) \cap X_p = \varnothing$, $\text{sg}(f(y_1))\,\sigma(y_1) = -1$ and p has at least $\nu + 1$ zeros in $[a, y_1]$ counting multiplicites up to order 2, where $[a, y_1] \cap S = \{z_1, \ldots, z_\nu\}$, to show that r_ϵ and f are copositive on $[a, y_1]$ for $\epsilon > 0$ and sufficiently small.

Thus, for the remainder of the proof, we shall assume that $w_i = 2$. In this case there exists $z_j \in [x_i, y_i] \cap S$ with $p'(z_j) = 0$ and we have that q vanishes at $s_i' < z_{i+1} < \cdots < z_{i+\nu} < s_i$ and only at these points in $[x_i, y_i]$, where we have assumed that $[x_i, y_i] \cap S = \{z_{i+1}, \ldots, z_{i+\nu}\}$. Now, in this case we must have $\text{sg}(f(x_i))\,\sigma(x_i) = \text{sg}(f(y_i))\,\sigma(y_i) = -1$. Since $\text{sgn}(q(x_i)) = \sigma(x_i)$ and $\text{sgn}(q(y_i)) = \sigma(y_i)$ we have that f and q are copositive on $[s_i', s_i]$. For the intervals $[x_i, s_i']$ and $[s_i, y_i]$ we have that p is never zero by construction, since $|f(x_i)| < |p(x_i)|$ and $|f(y_i)| < |p(y_i)|$. Thus, as before, for $\epsilon > 0$ sufficiently small, f and r_ϵ are copositive on these intervals since f and p are copositive there, and, hence, r_ϵ is copositive with f on $[x_i, y_i]$.

Finally, let us consider the cases where $x_1 > a$ or $y_\mu < b$. Without loss of generality we shall consider the case where $x_1 > a$. By the argument given in the $[y_i, x_{i+1}]$ case, it follows that $\max\{|f(x) - r_\epsilon(x)| : x \in [a, x_1]\} < \|f - p\|$, for $\epsilon > 0$ sufficiently small. We will now show that there exists $\alpha_1 > 0$ such that for $0 < \epsilon \leqslant \alpha_1$, r_ϵ is copositive with f on $[a, x_1]$. First of all, if $\text{sg}(f(x_1))\,\sigma(x_1) = -1$, then p can have only simple zeros at $\{z_1, \ldots, z_\nu\} = [a, x_1] \cap S$ and no other zeros, since k is maximal. In this case, by resorting to local Taylor expansions of order 1 about each $z_j \in [a, x_1] \cap S$, we can show that r_ϵ is copositive with f on $[a, x_1]$ for $\epsilon > 0$ sufficiently small.

This approach is necessary since, in this case, f and $-q$ are copositive on $[a, x_1]$, so that we must use the fact that p has no additional zeros (including multiplicities) to establish this case. On the other hand, if $\text{sg}(f(x_1))\,\sigma(x_1) = 1$, then q and f are copositive on $[a, x_1]$, so that f and r_ϵ are copositive for any $\epsilon > 0$. A similar argument applies to the interval $[y_\mu, b]$. From this it follows that r_ϵ is copositive with f on $[a, x_1]$ and $[y_\mu, b]$.

Combining all these cases, it follows that there exists $\epsilon > 0$ for which r_ϵ is copositive with f on $[a, b]$ and $\|f - r_\epsilon\| < \|f - p\|$. This is the desired contradiction and the proof of the theorem is complete.

If $p'(z_i) \neq 0$, $i = 1,\ldots, m$, then a simpler characterization theorem exists. Let $N = \{q \in M: q(z_i) = 0\}$, and note that N is an $(n - m)$-dimensional subspace of M.

THEOREM 2. *Suppose* $f \in C[a, b] \sim M$ *is admissible and* $p \in K_f$. *Let* $S = \{z_1,\ldots, z_m\}$ *and suppose that* $p'(z_i) \neq 0$, $i = 1,\ldots, m$. *Then the following are equivalent*:

(a) *p is a best copositive approximation to f.*

(b) *The zero element* $(0,\ldots, 0)$ *is in the convex hull of the set of* $(n - m)$-*tuples* $\{\sigma(x)\,\hat{x}: x \in X_p\}$, *where* $\hat{x} = (\phi_1(x),\ldots, \phi_{n-m}(x))$, *with* $\phi_1,\ldots, \phi_{n-m}$ *any basis for* N.

(c) *There exist* $n - m + 1$ *points in* X_p, $x_1 < x_2 < \cdots < x_{n-m+1}$, *such that* $\sigma(x_i)\,\pi(x_i) = (-1)^{i+1}\,\sigma(x_1)\,\pi(x_1)$, *where* $\pi(x_i) = \text{sgn} \prod_{j=1}^{m} (x_i - z_j)$.

Proof (a) $\Rightarrow$ (b). Let

$$\begin{aligned}
X_{+1} &= \{x \in [a, b]: f(x) - p(x) = \|f - p\|\},\\
X_{-1} &= \{x \in [a, b]: f(x) - p(x) = -\|f - p\|\},\\
X_{+2} &= \{x \in [a, b]: p(x) = 0, f(x) > 0\},\\
X_{-2} &= \{x \in [a, b]: p(x) = 0, f(x) < 0\},
\end{aligned}$$

The proof of this part is similar to the proof of the same part of Theorem 1 in [11], and will be omitted. Also, the proof of (b) $\Rightarrow$ (c) is similar to the proof of Theorem 3.1 in [5].

(c) $\Rightarrow$ (a). We will show that $f - p$ alternates once on each of the intervals (x_i, x_{i+1}), $i = 1, 2,\ldots, n - m$, so that, by Theorem 1, p is a best copositive approximation to f.

Let $[x_i, x_{i+1}] \cap S = \{z_{i+1},\ldots, z_{i+\nu_i}\}$, $\nu_i \geq 0$. Note that $\pi(x_i)/\pi(x_{i+1}) = (-1)^{\nu_i}$. Thus, from (c), $\sigma(x_2) = -\sigma(x_1)\,\pi(x_1)/\pi(x_2) = (-1)^{\nu_1+1}\,\sigma(x_1)$, so that $f - p$ alternates once on (x_1, x_2). Similarly, $\sigma(x_3) = \sigma(x_1)\,\pi(x_1)/\pi(x_3) = \sigma(x_1)(\pi(x_1)/\pi(x_2))(\pi(x_2)/\pi(x_3)) = -\sigma(x_2)\,\pi(x_2)/\pi(x_3) = (-1)^{\nu_2+1}\,\sigma(x_2)$. Thus, $f - p$ alternates once on (x_2, x_3). Continuing in this fashion, we see that

$f - p$ alternates once on each of the intervals (x_i, x_{i+1}), $i = 1,..., n - m$, so that $\{(x_i, x_{i+1})\}_{i+1}^{n-m}$ forms an alternant of length $n - m$. Hence, p is a best copositive approximation to f.

Remark. It is worth noting that the alternation result in part (c) of Theorem 2 is of a hybrid nature. That is, it contains elements of the alternation theorems for restricted range approximation and approximation with interpolatory constraints. The former enters in the sense that points in X_{+2} and X_{-2} are counted as extreme points, while the latter behavior is exhibited by the fact that the points where f changes sign act as nodes of interpolation. See [2, 5, 10] and remarks in [4, p. 212] for a comparison of the theorems and an elaboration of these points.

Let us now consider this problem for the case that the underlying space is a finite subset of $[a, b]$. Thus, let $X \subset [a, b]$ be a finite set of points with card $X \geqslant n + 1$. Further, let M be an n-dimensional Chebyshev subspace of $C[a, b]$. Fix $f \in C(X)$ and define K_f by $K_f = \{p \in M: p(x) f(x) \geqslant 0$ for all $x \in X\}$. In order to have a nontrivial problem we must restrict the behavior of f. Thus, let $X = \{x_i\}_{i=1}^{N}$, where $x_i < x_{i+1}$ for all i. We shall say that f changes sign at $x_i \in X$, $i = 1,..., N - 1$, if there exists $\nu > 0$, $i + \nu \leqslant N$, such that $f(x_i) f(x_{i+\nu}) < 0$ and $f(x_{i+1}) = \cdots = f(x_{i+\nu-1}) = 0$. (Note that if $\nu = 1$ then the second condition is not required.) In what follows, we shall require f to have no more than $n - 1$ sign changes. Define the sets U and L by

$$U = \{x \in X: f(x) > 0\}$$

and

$$L = \{x \in X: f(x) < 0\}.$$

Note that $U \cap L = \varnothing$, $L \cup U \cup \{x \in X: f(x) = 0\} = X$. Set $K_f = \{p \in M: p(x) \geqslant 0$ for all $x \in U$ and $p(x) \leqslant 0$ for all $x \in L\}$. Then this is a special case of restricted range approximation, for which a complete theory is known, including an alternation theorem [10]. Indeed, as before, we shall call $x \in X$ a positive extreme point for $f - p$ whenever $f(x) - p(x) = \|f - p\|_X$ or $x \in U$ and $p(x) = 0$($\|\cdot\|_X$ denotes the uniform norm on $C(X)$). Similarly, $x \in X$ is a negative extreme point whenever $f(x) - p(x) = -\|f - p\|_X$ or $x \in L$ and $p(x) = 0$. Set X_p equal to the union of all positive and negative extreme points. Then the alternation theorem is given by the next result, which follows from [10].

THEOREM 3. *$p \in K_f$ is a best copositive approximation to f if and only if there exist $n + 1$ points in X_p, $y_1 < y_2 < \cdots < y_{n-1}$, for which $\sigma(y_i) = (-1)^{i+1} \sigma(y_1)$, where $\sigma(y_i) = +1$ if y_i is a positive extremal and $\sigma(y_i) = -1$ if y_i is a negative extremal.*

Remark. Alternation theorems for approximation with various types of constraints have recently been obtained in [12, 13].

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